The elusive *Heisenberg Limit* in **QUANTUM ENHANCED METROLOGY**

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arXiv:1201.3940 (2012)

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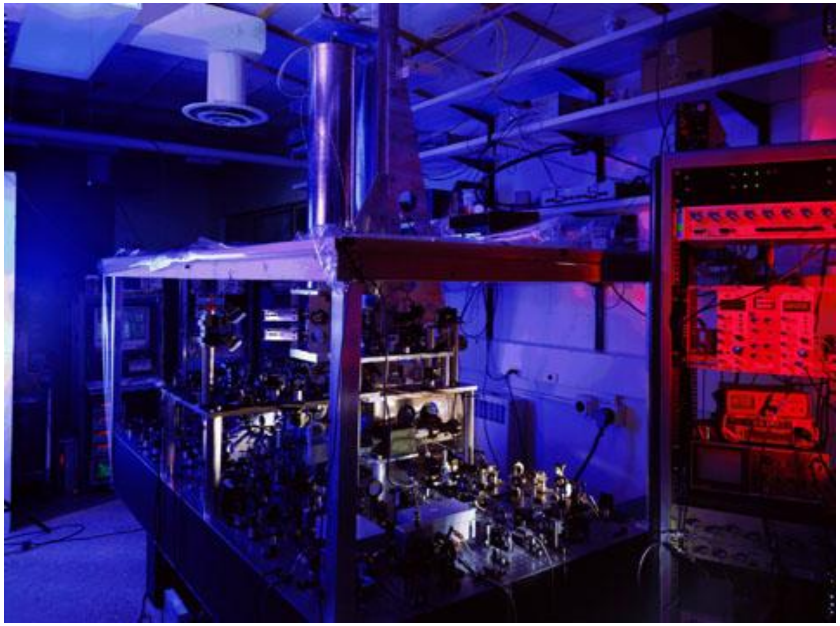
METROLOGY AT ITS (CLASSICAL) LIMITS

LIGO - gravitational wave detector

Michelson interferometer

$\Delta L/L \approx 10^{-22}$

NIST - Cs fountain atomic clock



Ramsey interferometry

$\Delta t/t \approx 10^{-16}$

PRECISION LIMITED BY:

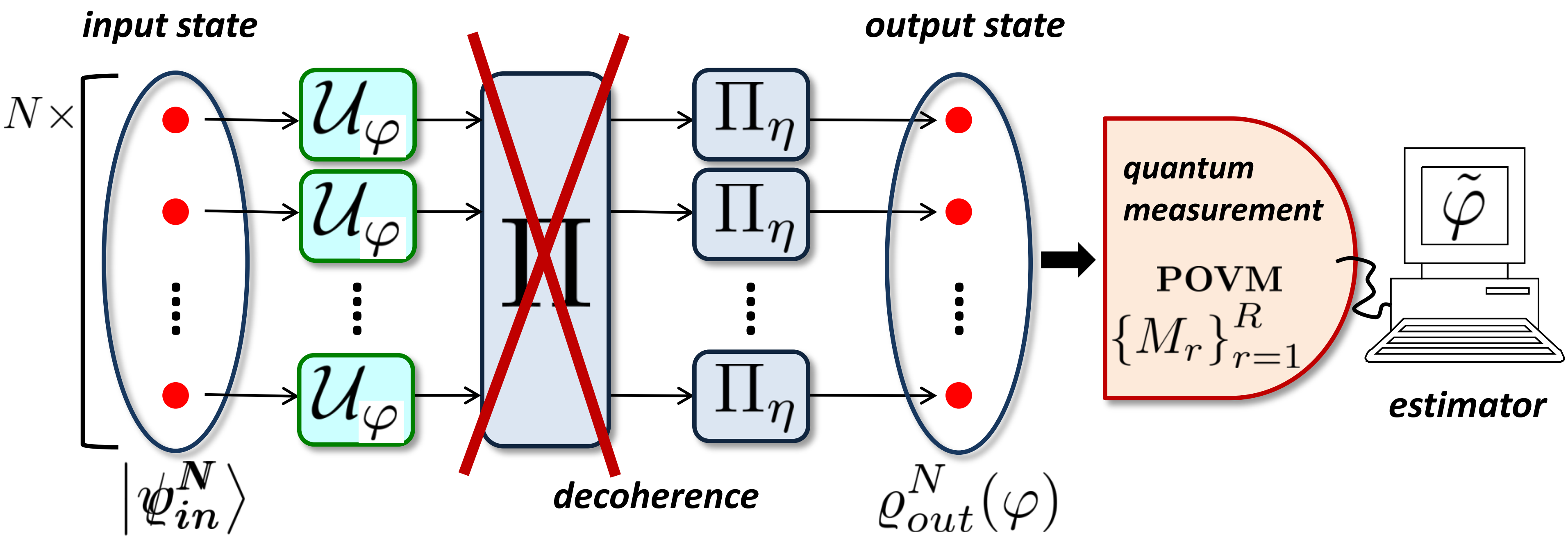
shot noise $\propto 1/\sqrt{N} = 1/N^{1/2}$

N- number of photons $\approx 10^{12}$/ns

projection noise $\propto 1/\sqrt{N} = 1/N^{1/2}$

N- number of atoms $\approx 10^{7}$

(Quantum) Estimation setup



"Aim / Rules of the Game":

- Minimise the average error: $\Delta\tilde{\varphi}=\sqrt{\left\langle(\tilde{\varphi}-\varphi)^2\right\rangle}$

  → Find the optimal method of establishing $\tilde{\varphi}$ as close to φ.
- Optimal to consider **pure input states**.
- Independent decoherence is most destructive → **ignore collective decoherence effects**.
- **Still hard!**, as we need to optimise over:

the input state + the set of all POVMs + the estimator.

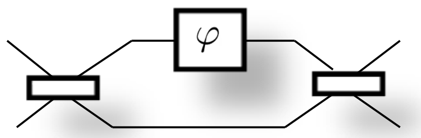
Upper (lower) bound on precision (error) by means of (Quantum) Fisher information

Cramer-Rao bound: $\Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}}$ *F* – Fisher information

(*depends only on the input state*)

Optical Interferometer

No decoherence



$$\left|\psi_{out}^{N}(\varphi)\right\rangle = \mathrm{e}^{-\mathrm{i}\frac{\varphi}{2}\left(\hat{n}_a^\dagger \hat{n}_a - \hat{n}_b^\dagger \hat{n}_b\right)} \left|\psi_{in}^{N}\right\rangle$$

- **Fisher Information easy to calculate.**
- **Optimal *N* photon state (maximal *F=N*^{2}):**

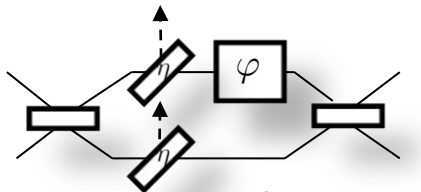
$$\left|\psi_{in}^{N}\right\rangle = \frac{1}{\sqrt{2}}\left(|N,0\rangle + |0,N\rangle\right)$$

Heisenberg Scaling

$$\boxed{\Delta\tilde{\varphi} \geq \frac{1}{N}}$$

J. J. . Bollinger, W. M. Itano, D. J. Wineland, and D. J. Heinzen, *Phys. Rev. A* **54**, R4649 (1996).

With decoherence



- **The output state is mixed.**
- **Fisher Information difficult to calculate.**
- **Optimal states do not have simple structure.**

R. Demkowicz-Dobrzanski et al,PRA **80**, 013825(2009), U. Dorner et al, PRL. **102**, 040403 (2009)

- **Asymptotic analytical lower bound:**

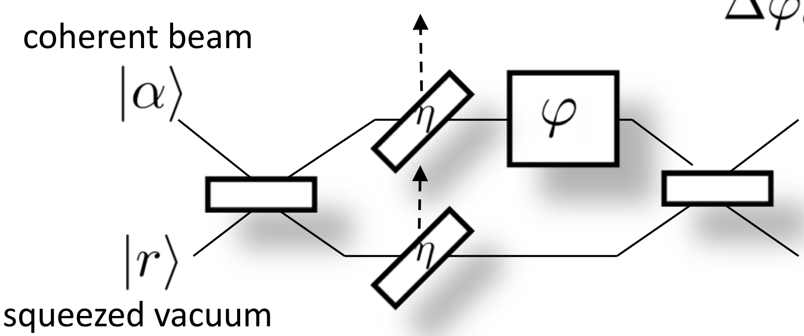
$$\boxed{\Delta\tilde{\varphi} \geq \sqrt{\frac{1-\eta}{\eta}\frac{1}{\sqrt{N}}}}$$

JK, R. Demkowicz-Dobrzanski, PRA **82**,053804 (2010), S. Knysh, V. Smelyanskiy, G. Durkin, PRA **83**, (2011)

ULTIMATE LOWER BOUND ON PRECISION BY MEANS OF (QUANTUM) FISHER INFORMATION

Cramer-Rao bound: $\Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}}$ *F* – Fisher information (***depends only on the input state***)

Saturable only for **infinite number of trials**, when estimating (**locally)** deviations from a known value of the parameter, φ_0. (*Other cases can only be worse* ☺)



$$\Delta\tilde{\varphi}_{\text{quantum}} \geq \sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}} \qquad \Delta\tilde{\varphi}_{\text{classical}} = \frac{1}{\sqrt{\eta N}}$$

$$\boxed{\frac{\Delta\tilde{\varphi}_{\text{quantum}}}{\Delta\tilde{\varphi}_{\text{classical}}} \geq \sqrt{1-\eta}}$$

LETTERS
PUBLISHED ONLINE: 11 SEPTEMBER 2011 | DOI: 10.1038/NPHYS2083
nature physics

$$\eta = 0.62$$

A gravitational wave observatory operating beyond the quantum shot-noise limit

The LIGO Scientific Collaboration †*

$$\frac{\Delta\varphi_{\text{squeezed}}}{\Delta\varphi_{\text{coherent}}} \approx 0.66 \qquad \frac{\Delta\tilde{\varphi}_{\text{quantum}}}{\Delta\tilde{\varphi}_{\text{classical}}} \geq 0.617$$

Heisenberg scaling is lost even for infinitesimal decoherence!!!

Is there a simpler, more general and more intuitive explanation?

Yes!!! – *almost all* decoherence models possess this property.

Answer realized via means of **two methods**:

Classical Simulation Method

- Stems from possibility to simulate **quantum chanels via classical probabilistic mixtures**.
- Optimal simulation corresponds to an **simple, intuitive, geometric representation**.
- Proves that ***almost all* (including full rank)** channels **asymptotically scale classically**.
- Allows to **derive a bound in 60 seconds** ☺.

Channel Extension Method

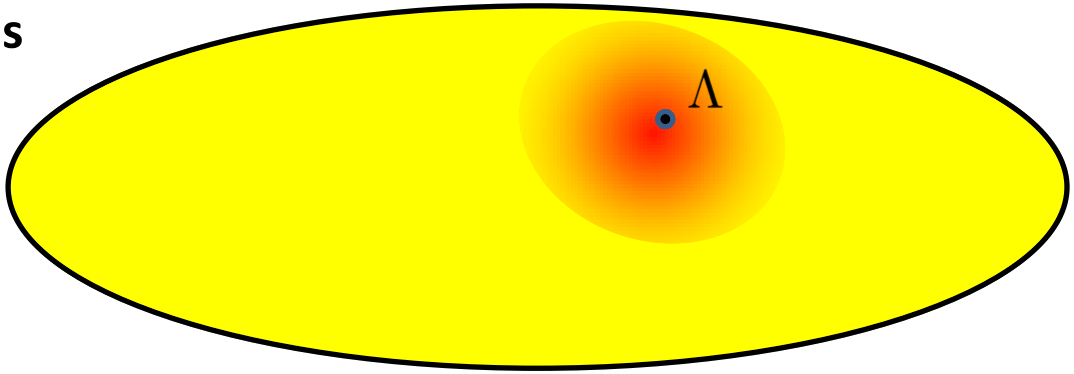
- Extends the CSM method to some **φ-extremal** channels.
- Sometimes provides **even tighter bounds** at the expense of the analyticity of solutions.
- However, the **bounds can always be efficently found numerically** by means of Semi-Definite Programming.

R.Demkowicz-Dobrzanski, JK, M. Guta, arXiv:1201.3940 (2012)

Classical simulation of a quantum channel

Convex set of quantum channels (CPTP maps)

$$\Lambda = \int \mathrm{d}X \, p(X) \Lambda_X$$

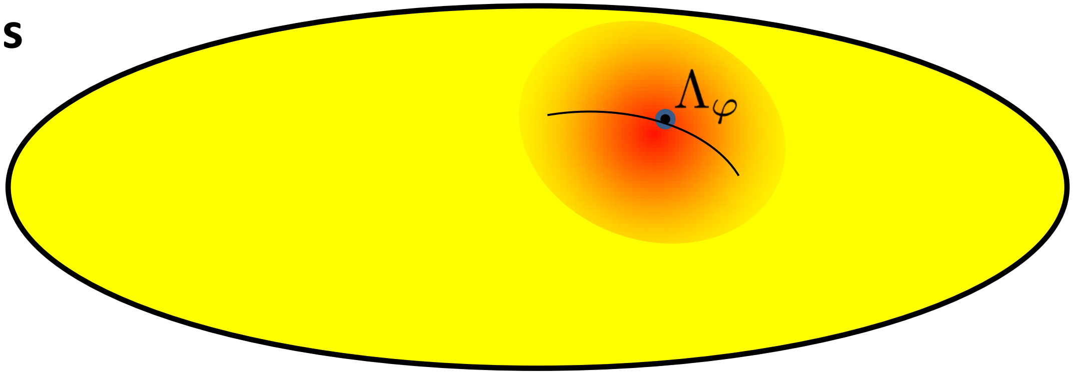


$$\Lambda : \ \varrho_{in} \in B\left(\mathcal{H}_{d_{in}}\right) \longrightarrow \varrho_{out} \in B\left(\mathcal{H}_{d_{out}}\right)$$

CLASSICAL SIMULATION OF A QUANTUM CHANNEL

Convex set of quantum channels (CPTP maps)

$$\Lambda_\varphi = \int \mathrm{d}X \, p_\varphi(X) \Lambda_X$$



Parameter dependence moved to mixing probabilities

Before:

$$\varphi \rightarrow \Lambda_\varphi^{\otimes N}\left[\varrho_{in}^N\right] \rightarrow \varrho_{out}^N(\varphi) \rightarrow \tilde{\varphi}$$

Now (sampling from X^N):

$$\varphi \rightarrow p_\varphi \rightarrow X^N \rightarrow \cancel{\otimes_{i=1}^N \Lambda_{X_i}[\varrho_{in}^N] \rightarrow \varrho_{out}^N(\varphi)} \rightarrow \tilde{\varphi}$$

$$|\wedge$$

$$\varphi \rightarrow p_\varphi \rightarrow X^N \rightarrow \tilde{\varphi}$$

By Markov property….

Estimating directly from X^N *is no worse than from measurement on* $\varrho_{out}^N(\varphi)$

CLASSICAL *N* INDEPENDENT VARIABLES !!

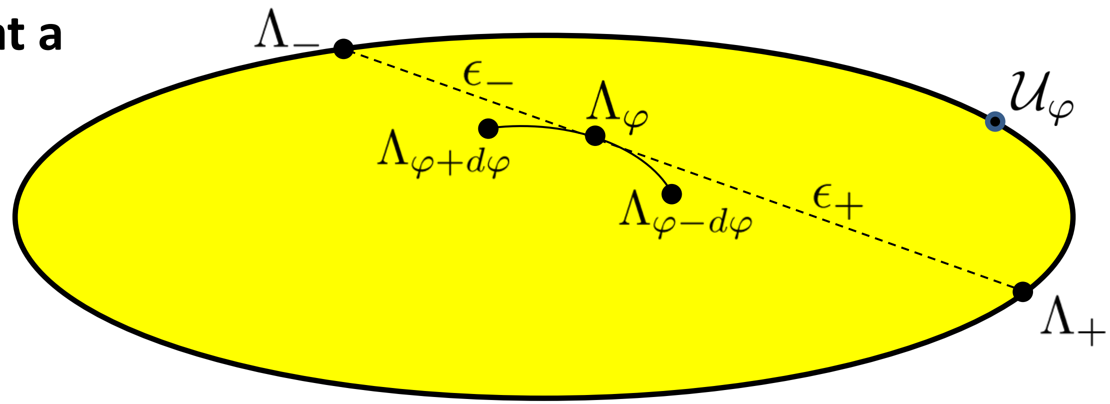
$$\boxed{F\left[\varrho_{out}^N(\varphi)\right] \leq N\, F_{\rm cl}[p_\varphi(X)]} \Longrightarrow \boxed{\Delta\tilde{\varphi} \geq \frac{1}{\sqrt{F}} \geq \frac{1}{F_{\rm cl}} \frac{1}{\sqrt{N}}}$$

K. Matsumoto, arXiv:1006.0300 (2010)

The "Worst" classical simulation

Quantum Fisher Information at a given φ depends only on

$$\Lambda_\varphi \qquad \partial_\varphi \Lambda_\varphi$$



It is enough to analize,,local classical simulation'':

$$\Lambda_\varphi = \int dX \; p_\varphi(X)\Lambda_X + O(d\varphi^2)$$

The „worst" classical simulation:

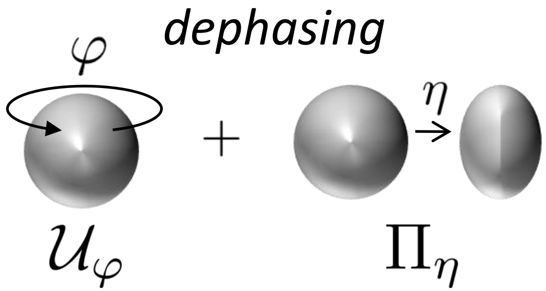
$$\Lambda_\varphi = p_+(\varphi)\Lambda_+ + p_-(\varphi)\Lambda_- + O(d\varphi^2) \qquad \Lambda_\pm = \Lambda_\varphi \pm \frac{d\Lambda_\varphi}{d\varphi}\epsilon_\pm$$

$$\boxed{\Delta\tilde{\varphi} \geq \sqrt{\frac{\epsilon_+\epsilon_-}{N}}}$$

Does <u>not</u> work for φ-extremal channels, e.g *unitaries* $\mathcal{U}_\varphi$.

R. Demkowicz-Dobrzanski, JK, M. Guta, arXiv:1201.3940 (2012)

QUBIT DEPHASING: DERIVATION OF THE BOUND IN 60 SECONDS!



$$\Pi_\eta \begin{cases} K_1 = \sqrt{\frac{1+\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ K_2 = \sqrt{\frac{1-\eta}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

$$\Lambda_\varphi[\rho] = \sum_i K_i \left(U_\varphi \rho U_\varphi^\dagger \right) K_i^\dagger$$

Choi-Jamiołkowski isomorphism (positive operators correspond to physical maps)

$$P_{\Lambda_\varphi} = \Lambda_\varphi \otimes \mathbb{1}(|\Phi\rangle\langle\Phi|) \qquad |\Phi\rangle = \sum_i |i\rangle \otimes |i\rangle$$

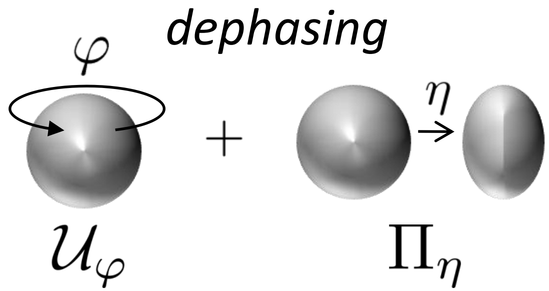
we look for $\varepsilon_\pm$ such that

$$P_{\Lambda_\varphi} \pm \varepsilon_\pm \partial_\varphi P_{\Lambda_\varphi} \geq 0$$

$$P_{\Lambda_\varphi} = \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi}\eta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\varphi}\eta & 0 & 0 & 1 \end{pmatrix}$$

R. Demkowicz-Dobrzanski, JK, M. Guta, arXiv:1201.3940 (2012)

Qubit Dephasing: derivation of the bound in 60 seconds!



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we look for $\varepsilon_\pm$ such that

$$P_{\Lambda_\varphi} \pm \varepsilon_\pm \partial_\varphi P_{\Lambda_\varphi} \geq 0$$

$$P_{\Lambda_\varphi} + \varepsilon \partial_\varphi P_{\Lambda_\varphi} = \begin{pmatrix} 1 & 0 & 0 & e^{i\varphi}\eta(1+i\varepsilon) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{-i\varphi}\eta(1-i\varepsilon) & 0 & 0 & 1 \end{pmatrix}$$

$$\eta^2(1+\varepsilon^2) \leq 1$$

$$\varepsilon \leq \frac{\sqrt{1-\eta^2}}{\eta}$$

$$\boxed{\Delta\tilde{\varphi} \geq \sqrt{\frac{\varepsilon_+ \varepsilon_-}{N}} = \frac{\sqrt{1-\eta^2}}{\eta} \frac{1}{\sqrt{N}}}$$

Exactly the same as the bound of

B. M. Escher, et al. Nature Physics, **7,** 406 (2011)

(*minimization over different Kraus representations*)

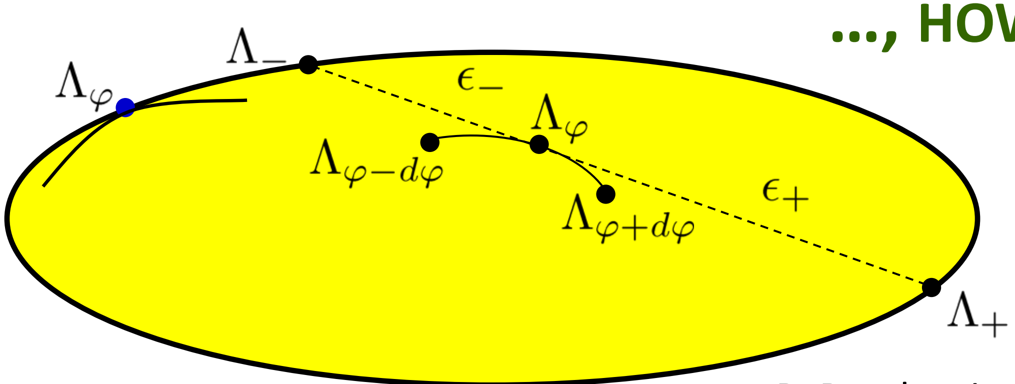
R. Demkowicz-Dobrzanski, JK, M. Guta, arXiv:1201.3940 (2012)

SUMMARY

- **Heisenberg Scaling** is **lost** for a generic decoherence channel even for *infinitesimal* noise.
- Simple ***bounds on precision*** can be derived using the *intuitive geometical picture* (*Classical Simulation Method*).
- φ**-extremal channels** (ones on boundary that is non-flat in the $\partial_\varphi \Lambda_\varphi$ direction) are **not** *classically simulable*.
- However, **such ones** (apart from *unitaries*) seem to be approachable by the *Channel Extension Method* and **scale classically**.

…but (*yet* ☺) **no disproof** that there is *no physical noise* that composed with free evolution still *allows* for **HS** asymptotic scaling!

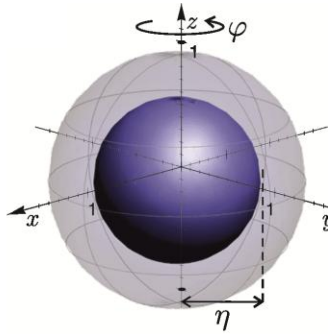
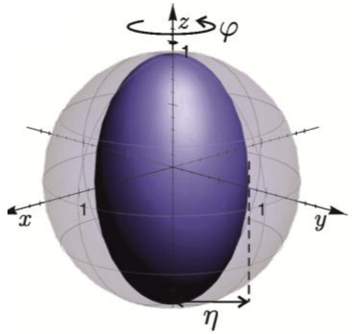
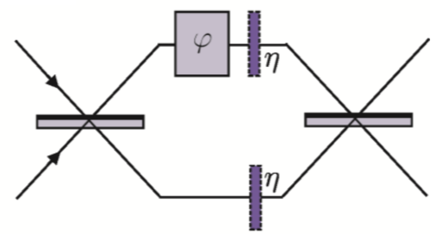
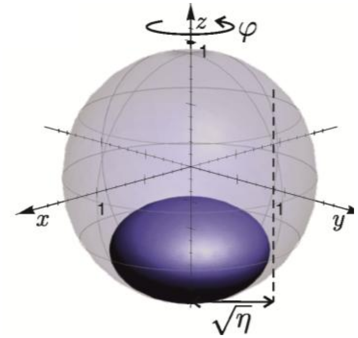
…, HOWEVER, WE STRONGLY DOUBT IT!



$$\Delta\tilde{\varphi} \geq \sqrt{\frac{\epsilon_+\epsilon_-}{N}}$$

R. Demkowicz-Dobrzanski, JK, M. Guta, arXiv:1201.3940 (2012)

GALLERY OF DECOHERENCE MODELS

<table>
<tr><th colspan="2">Depolarization</th><th colspan="2">Dephasing</th><th colspan="2">Lossy interferometer</th><th colspan="2">Spontaneous emission</th></tr>
<tr><td colspan="2">
</td><td colspan="2">
</td><td colspan="2">
</td><td colspan="2">
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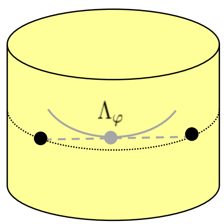
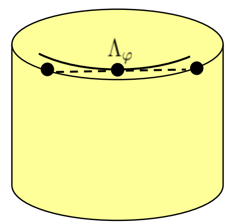
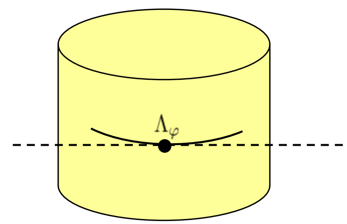
<tr><td colspan="2">

inside the set
of quantum channels
full rank</td><td colspan="2">

on the boundary,
non-extremal,
not φ-extremal</td><td colspan="2">

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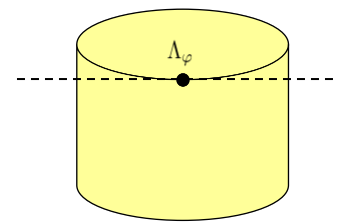
but φ-extremal</td><td colspan="2">

on the boundary,
extremal</td></tr>
<tr><td>CSM</td><td>$\sqrt{\frac{1+3\eta}{4\eta}\cdot\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$</td><td>CSM</td><td rowspan="2">$\frac{\sqrt{1-\eta^2}}{\eta}\frac{1}{\sqrt{N}}$</td><td>CSM</td><td>N/A</td><td>CSM</td><td>N/A</td></tr>
<tr><td>CEM</td><td>$\sqrt{\frac{1+2\eta}{2\eta}\cdot\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$</td><td>CEM</td><td>CEM</td><td>$\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$</td><td>CEM</td><td>$\frac{1}{2}\sqrt{\frac{1-\eta}{\eta}}\frac{1}{\sqrt{N}}$</td></tr>
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$$\Delta\tilde{\varphi} \geq \mathrm{bound}_{\mathrm{CEM}} \geq \mathrm{bound}_{\mathrm{CSM}}$$